Transposing, and substituting $\chi_1 I_1$ for $\int_0^P h_1 \, dp_1$ etc. we obtain

$$\frac{\frac{1}{2}(A_{P,1}+A_{P,2}) - \pi r_1^2 \left[1 + \frac{P}{E}(3\sigma - 1) - \frac{\delta r}{r}\right]}{\frac{1}{2}(A_{P,1}-A_{P,2}) - \pi r_1^2 \frac{\delta r}{r}} = -\frac{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}+1\right)}{\left(\frac{\chi_2}{\chi_1}\frac{I_2}{I_1}-1\right)}. \quad (5.5)$$

The right hand side of this equation is completely determined by χ_2/χ_1, the ratio of the cube roots of the measured rates of flow, and the ratio I_2/I_1. The quantity $(A_{P,1} - A_{P,2})$ occurring in the denominator of the left hand side may be determined by direct balancing of the two forms of the assembly against any third reference assembly, and the quantity $\delta r/r$ is established by diametral measurements on the two pistons. The small quantity $P(3\sigma - 1)/E$ is known with sufficient accuracy from the elastic constants of the material. Subject, therefore, to further examination of the factor I_2/I_1 equation (5.5) enables the quantity $(A_{P,1} + A_{P,2})/2$, i. e. the mean of the effective areas of the two forms of the assembly, to be determined, as a function of the applied pressure, from the experimental observations.

It is evident that the term $\pi r_1^2 \cdot \delta r/r$ occurring in the denominator of the left hand side should be identical with $(A_{0,1} - A_{0,2})$ and this may be checked directly from the experimental data. If, as may be the case, the difference $(A_{P,1} - A_{P,2})$ is independent of pressure, the denominator of the left hand side may be written more simply as $-(A_{0,1} - A_{0,2})/2$, but it cannot of course be assumed a priori that this condition will hold.

c) *Treatment of the integral 'I'*

In order to estimate the value of I some simplifying assumptions must be introduced if the theory is not to become unjustifiably complicated. From the second of equations (5.4) it is clear that we can calculate I if we can express h and η as functions only of p. As regards h, the justification for assuming that the part of $h(x)$ arising from distortion due to the pressure in the interspace between the piston and cylinder may be taken as proportional to the pressure $p(x)$ at the same position has already been discussed. Bearing in mind that we are not really interested in the absolute values of I_1 and I_2, but only in their ratio, this assumption is not likely to lead us far astray. As before, there is an additional component of h arising from the longitudinal thrust on the piston, which will be proportional to the total applied pressure, P. We therefore write

$$h = H + \nu P + \mu p \quad (5.6)$$

where μ and ν are constants.

The coefficient of viscosity at constant temperature is certainly determined uniquely by the pressure and there is considerable evidence available from published measurements that the dependence may be represented reasonably closely by an exponential function, in other words that we may write

$$\eta = \eta_0 \, e^{\alpha p} \quad (5.7)$$

where α is a constant and η_0 is the value at zero (or atmospheric) pressure. This relation has been found to hold with fair accuracy for most oils of types likely to be used in conjunction with pressure balances, although it appears that there may be more pronounced departures in the case of some silicone fluids (BRIDGMAN 1952; Amer. Soc. Mech. Engrs. 1953; ZOLOTYKH 1960).

The evaluation of I in terms of the constants in equations (5.6) and (5.7) is now straightforward and, writing for brevity

$$c = (H + \nu P)\, \alpha/\mu \ ,$$

we obtain

$$I = (l\eta_0)^{\frac{1}{3}} P^{\frac{3}{2}} I' \ ,$$

where

$$I' = (\alpha P)^{\frac{1}{3}} \left(c + \frac{\alpha P}{2}\right) \begin{Bmatrix} c^3 + 3c^2 + 6c + 6 \\ - e^{-\alpha P}[(c+\alpha P)^3 + 3(c+\alpha P)^2 + 6(c+\alpha P) + 6] \end{Bmatrix}^{-\frac{1}{3}} \quad (5.8)$$

This quantity may conveniently be represented as a family of graphs showing its dependence on $(H + \nu P)/\mu P$ for a suitable range of values of αP.

In order to apply equation (5.8) the values of $(H + \nu P)/\mu P$ and αP corresponding to the experimental points are required. Denoting by ϱ_P the ratio $\chi_2 I_2/\chi_1 I_1$ at a given applied pressure P, and by ϱ_0 the extrapolated value of ϱ_P corresponding to zero applied pressure, and using equation (5.6), we find that

$$\varrho_P = \frac{H_2 + \nu P + \mu P/2}{H_1 + \nu P + \mu P/2} \ ,$$

whence, after some reduction, we obtain the equations

$$\frac{H_1 + \nu P}{\mu P} = \frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2} \quad (5.9)$$

and

$$\frac{H_2 + \nu P}{\mu P} = \varrho_P \frac{\varrho_0 - 1}{\varrho_0 - \varrho_P}\left(\frac{\nu}{\mu} + \frac{1}{2}\right) - \frac{1}{2} \ . \quad (5.10)$$

We do not need to know the values of μ and ν, but an approximate figure for ν/μ is required. From the elementary theory leading to equation (2.6) we easily find

$$\nu/\mu = -\sigma/2 \text{ (approx.)}$$

whence we obtain, with sufficient accuracy, $\nu/\mu = -0.15$. Initially, of course, we cannot actually use the ratio $\chi_2 I_2/\chi_1 I_1$ since I_2/I_1 has not yet been determined. In practice, therefore, we commence by using simply the experimental ratio χ_2/χ_1 to obtain a first approximation to the correction factor, and then if necessary proceed to a second approximation.

To derive the appropriate value of αP we again ignore initially the distinction between $I_1(P)$ and $I_2(P)$, and denoting the quantity $(\chi_2 - \chi_1) P^{-\frac{1}{3}}$ at the applied pressure P by $\Delta\chi(P)$, we obtain from equations (5.4) and (5.6)

$$\Delta\chi(P) = P^{\frac{2}{3}}(H_2 - H_1)/I(P)$$
$$= (H_2 - H_1)(l\eta_0)^{-\frac{1}{3}}/I'(P)$$

whence, dividing the experimental value of $\Delta\chi(P)$ into the extrapolated value corresponding to zero applied pressure, we have

$$\Delta\chi(O)/\Delta\chi(P) = I'(P)/I'(O) \ .$$